Comment on P. K. Clark's Distribution of Lognormal-Normal Increments And the Lognormal Cascade Distribution

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Second Draft revised on Jan 19, 2010

Abstract

To illustrate that the mathematical form of P. K. Clark's distribution of lognormal-normal increments, with a minor bug fix, is the same as the symmetric lognormal cascade distribution that I studied in 2008-9, although the two were derived from very different stochastic assumptions.

1 Introduction

I have been studying the so-called Lognormal Cascade Distribution (Lihn, 2008, 2009, 2010). This distribution can be derived as the aggregate log-return distribution of a market consisting a group of stocks (Lihn 2009). It turns out such distribution is in the same form of the distribution proposed in P. K. Clark's 1973 paper. This is amazing since the methodology and thinking behind mine and that of Clark's are very different. Yet we reached the same static distribution. In this comment, I provides a quick proof that they are identical, using the symbolic algebra tool, GNU Maxima.

2 Change of Variables

In page 12 of P. K. Clark's 1973 paper, Theorem (5) proposed that a random process subordinated to a normal process with independent increments distributed $N(0, \sigma_2^2)$ and directed by a lognormal with independent increments (and parameters μ and σ_1^2) has the following lognormal-normal increments:

$$f_{\mathrm{LNN}}(y) = \frac{1}{2\pi\, \sigma_1^2\, \sigma_2^2} \int_0^\infty v^{-\frac{3}{2}} e^{-\frac{(\log\, v-\mu)^2}{2\sigma_1^2}} e^{-\frac{y^2}{2v\sigma_2^2}} dv\ . \tag{1}$$

This original formula has a minor bug that its integral does not yield to one (This can be verified quickly via GNU Maxima, see Section 4):

$$\int_{-\infty}^{\infty} dy\, f_{\mathrm{LNN}}(y) = \frac{1}{\sigma_1 \sigma_2}. \tag{2}$$

The correct form should be:

$$f_{\mathrm{LNN}}^{\mathrm{fix}}(y) = \frac{1}{2\pi\, \sigma_1\, \sigma_2} \int_0^\infty v^{-\frac{3}{2}} e^{-\frac{(\log\, v-\mu)^2}{2\sigma_1^2}} e^{-\frac{y^2}{2v\sigma_2^2}} dv\ . \tag{3}$$

By change of variables, it can shown that $f_{\mathrm{LNN}}^{\mathrm{fix}}(y)$ is the same as the first-order symmetric lognormal cascade distribution:

$$\mathrm{pdf}(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\, \eta\, \sigma(z)} e^{-\frac{z^2}{2\eta^2}} e^{-\frac{x^2}{2\sigma(z)^2}}, \tag{4}$$

where $\sigma(z) = \Phi e^z$ and η, Φ are parameters defining the distribution $\mathrm{pdf}(x)$.

The required change of variables are:

$$\begin{aligned} v &= e^{2z+\mu}, \\ \sigma_1 &= 2\eta, \\ \sigma_2 &= \Phi e^{-\mu/2}. \end{aligned} \tag{5}$$

Notice that, in Clark's formulation, μ and σ_2 are redundant variables.

3 Implication

The implication of equivalent mathematical form may be significant. The result of a subordinated process may be expressed alternatively by a group of constituents with varying population distribution characteristics. This could be a new research direction for what will become an important field of fat tailed process, which the financial systems have exhibited clearly.

4 Appendix: Maxima Program

The Maxima program that performs the change of variables is outlined below. Using Maxima reduces chance of errors as opposed to carrying out the algebra on paper. The integrand of Equation (3) is defined by LNN as following:

```
LNN_1 : v^(-3/2)/(2*%pi*s1*s2);
            /* LNN_1: Both squares on s1 and s2 are removed */
LNN_2 : %e^( -(log(v)-mu)^2/(2*s1^2) );
LNN_3 : %e^(-y^2/(2*v*s2^2));
LNN : LNN_1 * LNN_2 * LNN_3;
```

We can validate that $\int_{-\infty}^{\infty} dy\, f_{\mathrm{LNN}}^{\mathrm{fix}}(y) = 1$ by

```
assume(s1>0, s2>0, mu>0, v>0 );
integrate(integrate(LNN, y, minf, inf), v, 0, inf);
      /* answer: 1 */
```

To perform the change of variables on Equation (3) to yield Equation (4), the following steps are taken:

```
assume(phi>0);
v_sub : %e^(2*z+mu);
dv_factor : diff(v_sub, z);
s1_sub : 2*eta;
s2_sub : phi*%e^(-mu/2);
pdf : factor(ev(LNN*dv_factor, v=v_sub, s1=s1_sub, s2=s2_sub));
```

The output of pdf is

```
%e^(-(y^2*%e^(-2*z))/(2*phi^2)-z^2/(2*eta^2)-z)/(2*%pi*eta*phi)
```

This is exactly the same as Equation (4).

References

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